

## AN INTRODUCTION TO GAUGE THEORIES

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### ABSTRACT

These lecture notes present an introduction to gauge theories: the systematics of Yang-Mills theories, spontaneous symmetry breaking, and Higgs mechanism. The treatment is simple, stressing the general principles rather than detailed calculations. We present the Weinberg-Salam model as an example of a renormalizable theory of weak and electromagnetic interactions of leptons, and we show that the extension of these ideas into the hadronic world requires the introduction of charm and colour. Finally, we try to include strong interactions into the scheme, guided by the experimental results of deep-inelastic lepton-nucleon scattering. We derive and solve the Callan-Symanzik equation, and we introduce the concepts of asymptotic freedom and quark confinement.

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### FOREWORD

These are the notes of the lectures I gave at the 1977 CERN School of Physics. They are not a review article! Therefore, they do not claim to exhaust a subject which kept busy most high-energy theorists for the last years. They mainly contain the basic ideas of gauge theories, namely the systematics of Yang-Mills theories and the phenomenon of spontaneous symmetry breaking. These subjects have already been exposed in several excellent review articles or lecture notes, so this report is addressed primarily to those who have followed my lectures, as a supplement to their own notes. No attempt has been made to describe detailed models of gauge theories or to compare them with the recent experimental results. The emphasis was on general principles, not specific applications. Finally, since the audience was formed mainly by experimentalists, all the technical part of gauge theories (quantization, Feynman rules, renormalization, etc.) has been omitted and several theoretical arguments have been, often dangerously, oversimplified.

## 1. INTRODUCTION

The idea of unifying weak and electromagnetic interactions is very old and goes back to the classical work of Fermi. On the phenomenological level the two forces present some common features, but also several important differences. On the theoretical level, quantum electrodynamics was, for years, the only consistent and successful theory we had in elementary particle physics. It provided the archetype for any other physical theory. It is essentially based on a very general technique, called perturbation theory. Let  $H$  be the Hamiltonian describing the dynamics of a physical system. In principle, we would know everything about the system, if we could solve the eigenvalue problem:

$$H \psi_n = E_n \psi_n, \quad (1.1)$$

where  $E_n$  are the eigenvalues and  $\psi_n$  the eigenfunctions of  $H$ . Unfortunately, the exact solution of (1.1) is unknown for all but some very simple systems, and in practice we are obliged to use some kind of approximation schemes. Perturbation theory amounts to splitting  $H$  into two parts:

$$H = H_0 + \lambda H_I \quad (1.2)$$

where  $H_0$ , called the "unperturbed part of  $H$ " is a Hamiltonian chosen so that the solution of the eigenvalue problem is known exactly and  $\lambda H_I$  is called the "perturbation";  $\lambda$  is some parameter which characterizes the strength of the perturbation. The idea is to find a splitting such that  $\lambda H_I$  is a relatively small part of  $H$ . Then, we solve the eigenvalue problem of  $H_0$  and calculate the corrections on the eigenvalues and eigenfunctions, induced by the presence of the perturbation, as a series in powers of  $\lambda$ . Theoretically, there may be more than one way to obtain such a splitting of the total Hamiltonian, leading to more than one possible perturbation expansion, but in practice the choice is very limited. In a relativistic quantum field theory in four space-time dimensions, the only eigenvalue problem which is always exactly solvable is that of a free field theory, i.e. if  $H_0$  describes a system of free particles. Consequently, we are obliged to include in the perturbation part  $\lambda H_I$ , the entire interaction Hamiltonian. Furthermore, the complexity of the calculations is such that we can only compute the first few terms in the power series expansion. Obviously, such a scheme has some chances to give sensible results only if the entire interaction is "weak". This means, physically, that the energy due to the interactions must be small

compared to the kinetic energies and masses of the particles which are included in  $H_0$ . Technically this is translated to the requirement that  $\lambda$ , the "strength" of the interaction, is a small number,  $\lambda \ll 1$ . In quantum electrodynamics this parameter is the fine structure constant  $\alpha$  which equals  $1/137$ . This small number is responsible for the practical successes of QED because successive terms in the perturbation expansion are proportional to increasing powers of  $\alpha$ , so they get smaller and smaller, and a good approximation is obtained by keeping only the first few of them.

Given this great success, we are naturally tempted to apply the same method to the other interactions. Unfortunately this is not straightforward. In strong interactions we can still, formally, write the analogue of Eq. (1.2), but the corresponding parameter  $\lambda$  turns out to be large,  $\lambda \gtrsim 1$ , and the approximation scheme breaks down. This means physically that, for a system of hadrons, the energy due to their strong interactions is not a small part of their total energy. Then what about weak interactions? We know, experimentally, that they are indeed weaker than the electromagnetic ones and, therefore, we expect here perturbation theory to give even better results. But we now face a different problem: as everybody knows, calculations in quantum field theory are not so simple. The Feynman rules, if applied blindly, give meaningless results for all but the lowest-order terms, because all higher terms turn out to be divergent. Perturbation theory must be supplemented with a well-defined algorithm, called "renormalization theory", whose purpose is to extract meaningful finite answers out of the divergent expressions of the perturbation expansion. This algorithm had been invented in order to be applied to QED, and it turned out that it did not apply to the theory that described, phenomenologically, the weak interactions. Several people have tried, without success, to invent a new algorithm adapted to the Fermi theory of weak interactions, and finally the solution came from the opposite direction: the algorithm remained the same, but the phenomenological theory was replaced by a different one. The remarkable thing is that this new theory, which is far more beautiful from the aesthetic point of view, has different experimental consequences, and it now seems that they are verified. It looks like the old prejudice, that the search for internal consistency and aesthetic beauty always leads to a deeper understanding of the physical world, is once more confirmed.

## 2. PHENOMENOLOGY

Weak interaction phenomena were, until 1971, well described by a simple phenomenological model involving an operator  $J_\lambda(x)$ , the "weak current", which is the analogue of the familiar electromagnetic current,

$$\mathcal{L}_F = \frac{G}{\sqrt{2}} J_\lambda^\dagger(x) J_\lambda(x) . \quad (2.1)$$

This is the famous current  $\times$  current theory and  $G/\sqrt{2}$  is the Fermi coupling constant, which is equal to  $10^{-5} m_{\text{proton}}^{-2}$ . The weak current  $J_\lambda(x)$  is a sum of two parts, a leptonic part  $\ell_\lambda(x)$  and a hadronic one  $h_\lambda(x)$ . They are both of the V-A form and satisfy simple algebraic relations. The leptonic current can be written in terms of the fields of known leptons as:

$$\ell_\lambda(x) = \bar{\mu}(x) \gamma_\lambda (1 + \gamma_5) \nu_\mu(x) + \bar{e}(x) \gamma_\lambda (1 + \gamma_5) \nu_e(x), \quad (2.2)$$

where we have used the letters  $\mu$ ,  $e$ ,  $\nu_\mu$ , and  $\nu_e$  to denote the field operators of the corresponding particles. No simple expression exists for  $h_\lambda(x)$  in terms of the fields of known hadrons. It may take several forms depending on which particles we consider as elementary. What is more important, from the phenomenological point of view, is to know the general structure and the symmetry properties of  $h_\lambda(x)$ . It was a very important discovery -- which took several years and is due to the work of several people -- when it was finally established that the weak hadronic current  $h_\lambda(x)$  can be identified with the currents of the chiral symmetry group  $SU(3) \times SU(3)$  of the strong interactions. These properties can most easily be exhibited, for pure illustrative purposes, in a simple quark model. Let  $p(x)$ ,  $n(x)$ , and  $\lambda(x)$  represent the fields of the three quarks; then  $h_\lambda(x)$  is given by:

$$h_\lambda^\dagger(x) = \bar{p}(x) \gamma_\lambda (1 + \gamma_5) [\cos \theta n(x) + \sin \theta \lambda(x)], \quad (2.3)$$

where  $\theta$  is the Cabibbo angle.

Let me emphasize at this point that, despite its phenomenological character, (2.1) is an elegant structure that is rarely found in elementary particle physics. This simple and compact form could not only fit a large variety of data, but also it incorporated the physical principles of CVC, universality and algebraic properties that we mentioned above. Thus, at the phenomenological level, we had a perfectly working scheme; there was no compelling experimental reason to try to change it. It described correctly all experimental results which were inside its natural domain, namely all data which could definitely be attributed to weak interactions. In other words, if we could only forget for a moment that there was not much of a theory after all, and that the whole structure was just a

phenomenological description of the data, we would have every reason to be satisfied -- especially if we compared it with the situation in strong interactions, in which there was also no consistent theory, but there was no elegance either. And yet we were not happy! What we wanted was not a phenomenological scheme, but a physical theory. Yang-Mills theories were studied not because they fitted the data better, but rather because of their aesthetic beauty. As a matter of fact, at first -- and to a lesser extent even today -- it looks as though we had to pay a high price for the elegance they were giving us.

As we said in the Introduction, the model out of which we were getting our inspiration was QED. The first step towards unification was the intermediate vector boson hypothesis. The Lagrangian (2.1) was replaced by:

$$\mathcal{L}_w = g J^\lambda(x) W_\lambda(x) + \text{h.c.}, \quad (2.4)$$

where  $W_\lambda(x)$  is the field of a charged vector boson, which in weak interactions is supposed to play the role that the photon plays in QED. The relation between the semi-weak coupling constant  $g$  and  $G$  of Eq. (2.1) is

$$\frac{G}{\sqrt{2}} \approx \frac{g^2}{m_w^2} \quad (2.5)$$

with  $m_w$  being the  $W$  mass.

Until quite recently (2.4) described all weak interaction phenomena with only a charged current, but, as you all know, neutral weak currents have been discovered, so we need today at least three intermediate vector bosons with charges  $+$ ,  $-$ , and zero. Actually, the neutral currents arise naturally in gauge theories (you need special ingenious constructions, such as the Georgi-Glashow model, to make them disappear), and therefore they provided the first strong encouragement that with gauge theories we were on the right track. We had been going through many lean years in particle physics, and we are still amazed by the idea that, triggered off by purely theoretical considerations, we came across a great experimental discovery.

The Lagrangian (2.4) now looks very similar to QED since they both describe the interaction of a vector boson with an appropriate current. However, there are some important differences which cause the renormalization program to work for QED but not for (2.4). From the physical point of view these differences are the following.

- i) The electromagnetic interactions have a long range -- the photon is massless. Weak interactions give rise to short-range forces -- the intermediate vector bosons, if they exist at all, must be very massive ( $m_W \gtrsim 10-15 \text{ GeV}$ ).
- ii) The electromagnetic current is conserved, the weak current is not ( $\partial_\lambda J^\lambda(x) \neq 0$ ).
- iii) The photon is neutral. The W's come in three charge states.

These physical differences imply several technical ones, the most important of which concerns the vector boson propagators. We know that the photon propagator is given, in the Feynman gauge, by  $g_{\mu\nu}/k^2$ , and therefore behaves, at large momenta, like  $k^{-2}$ . The W-propagator, however, because of its non-zero mass, is given by

$$\frac{1}{k^2 - m_W^2} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2} \right),$$

which goes asymptotically like a constant. Therefore, if we look at fermion-fermion scattering through the exchange of two vector bosons, i.e. the diagrams of Fig. 1,

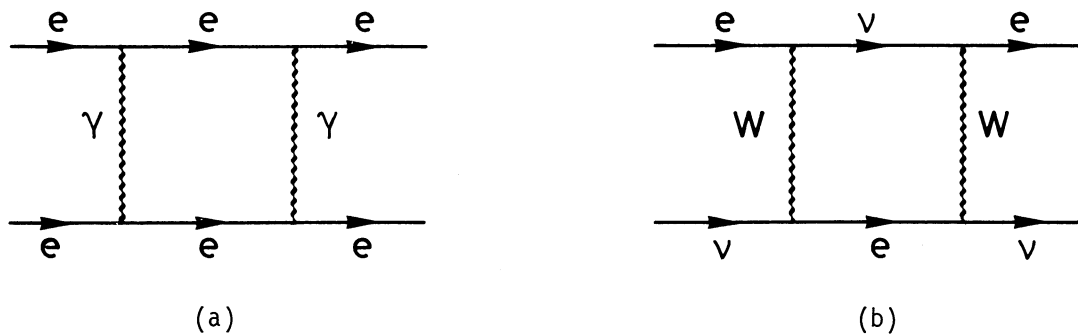


Fig. 1

we obtain, for the QED diagram (a), an integral which behaves, for large loop momentum, like:

$$\int \frac{d^4 k}{k^6} \rightarrow \text{finite},$$

while for diagram (b) the corresponding expression is

$$\int \frac{d^4 k}{k^2} \rightarrow \text{quadratically divergent}$$

because we no longer have the two  $k^{-2}$  convergence factors of the two photon propagators. Therefore, the Lagrangians (2.4) or (2.1), were uniquely used at the lowest possible order of perturbation theory and no higher-order corrections could be calculated. They were purely phenomenological objects.

Let us now try to solve the following seemingly hopeless problem: Is it possible to modify the Lagrangian (2.4) in such a way that we obtain a renormalizable field theory without upsetting its nice agreement with experiment? We all know that the answer is Yes and, in fact, we can go a long way towards the correct theory by a careful study of the one-loop diagrams. We will skip this part and apply instead an old theoretical prejudice which says that the best behaving theory is the most symmetric one. Notice that this was also the way things happened historically. The most symmetric way to couple neutral and charged vector bosons together is given by the Yang-Mills theories, which we shall study next.

### 3. GAUGE SYMMETRIES

At the basis of every symmetry principle in physics there is an assumption that some quantity is not measurable. For example, the assumption that there is no absolute position in space leads to the invariance under translations. Here we shall be interested in internal symmetries, i.e. transformations which do not affect the space-time point  $x$ . A simple example is given by the Lagrangian density of a free fermion field  $\psi(x)$ :

$$\mathcal{L}_0 = \bar{\psi}(x) (i \not{\partial} - m) \psi(x), \quad (3.1)$$

which is invariant under the phase transformation

$$\psi(x) \rightarrow e^{i\theta} \psi(x) \quad (3.2a)$$

$$\partial_\mu \psi(x) \rightarrow e^{i\theta} \partial_\mu \psi(x), \quad (3.2b)$$

where  $\theta$  is an arbitrary,  $x$ -independent phase. Formula (3.2b) follows from (3.2a), i.e. the derivative of the field transforms like the field itself. From Noether's theorem, the invariance under (3.2a) implies the existence of a conserved current of the form:

$$j_\mu(x) = \bar{\psi}(x) \gamma_\mu \psi(x) \quad ; \quad \partial^\mu j_\mu(x) = 0. \quad (3.3)$$

The group of transformations (3.2) is the Abelian group U(1).

We know in physics internal symmetries based on different groups. For example, invariance under isospin transformations is based on the group SU(2). The assumption is that the proton and the neutron are two different states of the same particle, the nucleon, and they transform into each other by isospin rotations, in the same way that the two spin states  $\pm\frac{1}{2}$  of a proton transform into each other by ordinary rotations. Similarly, we have the eightfold-way symmetry, based on the group SU(3), or the charm scheme, based on SU(4), etc.

In general, let  $\phi^i(x)$ ,  $i = 1, \dots, n$ , be a set of fields and  $\mathcal{L}_0[\phi^i(x), \partial_\mu \phi^i(x)]$  a Lagrangian density describing the dynamics of the system. An internal symmetry is an invariance of  $\mathcal{L}_0$  under a group G of transformations acting on the fields  $\phi^i(x)$ :

$$\phi^i(x) \longrightarrow \phi^i(x) + \theta^\alpha (T_\alpha)^i_j \phi^j(x) \quad ; \quad \begin{matrix} i=1, \dots, N \\ \alpha=1, \dots, N \end{matrix} \quad (3.4a)$$

where for future convenience we have written the infinitesimal transformations, i.e. we have kept only the first-order terms in an expansion in powers of  $\theta^\alpha$ .

The notation in (3.4) is:

N is the number of generators of G, i.e. the dimension of the associated Lie algebra. It equals 1 for the U(1) transformations (3.2), 3 for SU(2), 8 for SU(3), etc.

$T_\alpha$  are the matrices of the representation in which the fields  $\phi^i(x)$  belong. For example, if  $G = \text{SU}(2)$  and the  $\phi^i$ 's form an isodoublet,  $T_\alpha = i\tau_\alpha$ ,  $\tau_\alpha$  being the familiar Pauli matrices. If the fields form an isovector, then  $(T_\alpha)^i_j = \epsilon_{\alpha ij}$ , etc.

$\theta^\alpha$  are N c-number, infinitesimal, x-independent parameters.

Since the parameters  $\theta^\alpha$  are x-independent, the derivatives of the fields transform like the fields themselves:

$$\partial_\mu \phi^i(x) \longrightarrow \partial_\mu \phi^i(x) + \theta^\alpha (T_\alpha)^i_j \partial_\mu \phi^j(x). \quad (3.4b)$$

Such transformations, with x-independent parameters, will from now on be called "global" transformations<sup>\*)</sup>. Using again Noether's theorem we easily obtain, as a consequence of the invariance of  $\mathcal{L}_0$  under (3.4), N conserved currents.

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\*) They are sometimes called "gauge transformations of the first kind". I shall not use this terminology here, because I want to reserve the name "gauge" for the transformations that will be introduced in the sequel.



Let us go back to the simple Abelian example. The invariance of  $\mathcal{L}_0$  under (3.2) means that the phase of the field  $\psi(x)$  is not measurable, therefore it can be chosen arbitrarily. On the other hand, since it is  $x$ -independent, it must be chosen the same over the entire universe and for all times. This situation is clearly unsatisfactory on physical grounds. We would like instead to have a formalism which would allow us to fix the phase locally in a region with the dimensions of our experiment, without reference to far-away distances; in other words we would like to replace (3.2a) by

$$\psi(x) \rightarrow e^{i\theta(x)} \psi(x), \quad (3.5a)$$

where  $\theta$  is now a function of  $x$ . I want to emphasize here that this requirement is based on purely aesthetic arguments. If we adopt (3.5a) as a symmetry transformation, we face a serious problem, because now (3.2b) is replaced by:

$$\partial_\mu \psi(x) \rightarrow e^{i\theta(x)} \partial_\mu \psi(x) + i e^{i\theta(x)} \psi(x) \partial_\mu \theta(x), \quad (3.5b)$$

i.e. the derivative of the field no longer transforms like the field itself and, as a result, the Lagrangian (3.1) is no longer invariant under (3.5). We shall call transformations of the form (3.5), i.e. with  $x$ -dependent parameters, "local" or "gauge" transformations\*).

In differential geometry there is a standard way of restoring invariance under (3.5). Since the trouble arises from the derivative operator, we must introduce a new "derivative"  $D_\mu$ , called the "covariant derivative", which is again a first-order differential operator, but with the property that it transforms under (3.5a) like the field itself:

$$D_\mu \psi(x) \rightarrow e^{i\theta(x)} D_\mu \psi(x), \quad (3.5c)$$

In order to find such a  $D_\mu$ , we first introduce the affine connection which, in our language, is related to the "gauge field"  $A_\mu(x)$  and which, by definition, transforms like:

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \theta(x) \quad (3.6)$$

with  $e$  a constant.

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\*) They are also called "gauge transformations of the second kind".

We then define  $D_\mu$  by

$$D_\mu \equiv \partial_\mu + ie A_\mu, \quad (3.7)$$

and it is easy to verify that  $D_\mu \psi(x)$  does transform under (3.5a) and (3.6), as does (3.5c). Invariance under gauge transformations is now restored by replacing  $\partial_\mu$  by  $D_\mu$  everywhere in  $\mathcal{L}_0$

$$\begin{aligned} \mathcal{L}_0 \rightarrow \mathcal{L}_1 &\equiv \bar{\psi}(x) (i \not{\partial} - m) \psi(x) = \\ &= \bar{\psi}(x) (i \not{\partial} - m) \psi(x) - e \bar{\psi}(x) \not{x}_\mu \psi(x) A^\mu(x) \end{aligned} \quad (3.8)$$

The Lagrangian (3.8) is invariant under (3.5a) and (3.6), and it contains the gauge field  $A_\mu(x)$ . If we want to interpret the latter as the field representing the photon, we must add to (3.8) a term corresponding to its kinetic energy. This term must be, by itself, gauge-invariant, and we are thus easily led to the final Lagrangian

$$\mathcal{L}_1 \rightarrow \mathcal{L}_2 \equiv \mathcal{L}_1 - \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x), \quad (3.9)$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \quad (3.10)$$

Equation (3.9) is nothing else but the familiar Lagrangian of quantum electrodynamics. We have obtained it by just imposing invariance under gauge transformations. A final remark is in order:  $\mathcal{L}_2$  does not contain a term proportional to  $A_\mu A^\mu$ , since such a term is not invariant under (3.6). In other words, gauge invariance forces the photon to be massless.

The same procedure can be applied to the non-Abelian transformations (3.4). Again for aesthetic reasons we want to replace (3.4) by a group of local transformations in which  $\theta^\alpha \rightarrow \theta^\alpha(x)$ :

$$\varphi^i(x) \rightarrow \varphi^i(x) + \theta^a(x) (T_a)^i_j \varphi^j(x) \quad ; \quad \begin{matrix} i=1, \dots, n \\ a=1, \dots, N. \end{matrix} \quad (3.11a)$$

The derivative  $\partial_\mu \varphi^i(x)$  now picks up an extra term:

$$\begin{aligned} \partial_\mu \varphi^i(x) \rightarrow \partial_\mu \varphi^i(x) + \theta^a(x) (T_a)^i_j \varphi^j(x) + \\ + (T_a)^i_j \varphi^j(x) \partial_\mu \theta^a(x). \end{aligned} \quad (3.11b)$$

This last term spoils the invariance of  $\mathcal{L}_0$ . The rule for restoring invariance is again the same: we first introduce N gauge fields  $W_\mu^a(x)$ , which transform like:

$$W_\mu^a(x) \rightarrow W_\mu^a(x) + f_{bc}^a W_\mu^c \theta^b + \frac{1}{g} \partial_\mu \theta^a(x) \quad (3.12)$$

with g a constant and  $f_{cb}^a$ , the structure constants of the group, given by

$$[T_a, T_b] = f_{ab}^c T_c. \quad (3.13)$$

Formula (3.12) is just the generalization to non-Abelian groups of the photon transformation law (3.6). Notice that for the group U(1), f vanishes. In terms of the gauge fields we now define a covariant derivative

$$D_\mu \varphi^i(x) \equiv \partial_\mu \varphi^i(x) - g (T_a)^i_j W_\mu^a(x) \varphi^j(x) \quad (3.14)$$

and we can verify, using (3.11a) and (3.12), that it transforms like the fields, namely:

$$D_\mu \varphi^i(x) \rightarrow D_\mu \varphi^i(x) + \theta^a(x) (T_a)^i_j D_\mu \varphi^j(x). \quad (3.11c)$$

Finally, everywhere in  $\mathcal{L}_0$  we replace  $\partial_\mu$  by  $D_\mu$ :

$$\mathcal{L}_0(\varphi^i(x), \partial_\mu \varphi^i(x)) \rightarrow \mathcal{L}_1(\varphi^i(x), D_\mu \varphi^i(x)). \quad (3.15)$$

$\mathcal{L}_1$  is invariant under (3.11a) and (3.12) if  $\mathcal{L}_0$  was invariant under (3.4a). We can further add a kinetic energy term for the gauge fields which again is determined by the requirement of gauge invariance:

$$\mathcal{L}_1 \rightarrow \mathcal{L}_2 \equiv \mathcal{L}_1 - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}, \quad (3.16)$$

$$G_{\mu\nu}^a(x) = \partial_\mu W_\nu^a(x) - \partial_\nu W_\mu^a(x) - gf_{bc}^a W_\mu^b W_\nu^c. \quad (3.17)$$

Notice that, unlike the photons, non-Abelian gauge fields are self-coupled through the  $G_{\mu\nu}^a G^{\mu\nu a}$  term which contains trilinear and quartic interaction terms. Notice also that the gauge fields have still zero mass since a  $W_\mu^a W^\mu a$  term is again non-invariant under (3.12). Since no massless vector bosons, other than the photon, are known in nature, it looks as if non-Abelian gauge symmetries have nothing to do with physics in general and weak interactions in particular.

#### 4. SPONTANEOUSLY BROKEN SYMMETRIES

##### 4.1 Introduction

The realization that a physical problem possesses a certain symmetry often simplifies its solution considerably. For example, let us calculate the electric

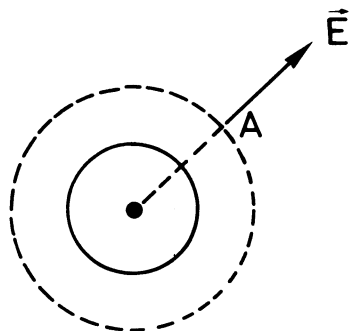


Fig. 2

field at a point A produced by a uniformly charged sphere (Fig. 2). One could solve the problem the hard way by considering the field created by a little volume element of the sphere and then integrating over. But any student knows that it is sufficient to realize that the problem has a spherical symmetry and then Gauss's theorem for the surface through A gives the answer immediately. In this reasoning we have implicitly assumed that symmetric problems always possess

symmetric solutions. Stated in this form the assumption sounds almost obvious; however, in practice, we need a much stronger one. Indeed, a real sphere is never absolutely symmetric and the charge is never distributed in a perfectly uniform way. Nevertheless, we still apply the above reasoning, hoping that small deviations from perfect symmetry will induce only small departures from the symmetric solutions. This, however, is a much stronger statement, which is far from obvious, since it needs not only the existence of a symmetric solution but also an assumption about its stability. And it is well known that this last property is not always true.

## 4.2 A simple classical example

A very simple counter example is provided by the problem of the bent rod.

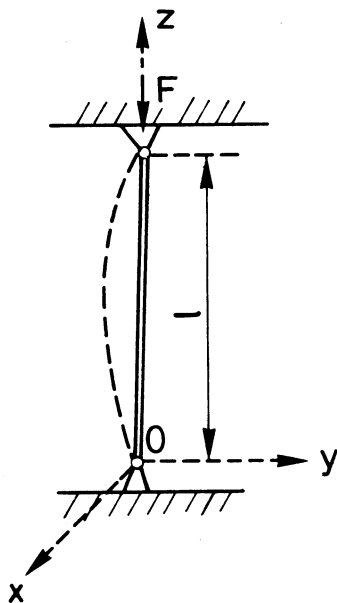


Fig. 3

Let a cylindrical rod be charged as in Fig. 3. The problem is obviously symmetric under rotations around the z-axis. Let  $z$  measure the distance from 0, and  $X(z)$  and  $Y(z)$  give the deviations, along the  $x$  and  $y$  directions respectively, of the axis of the rod at the point  $z$  from the symmetric position. For small deflections the equations of elasticity take the form:

$$IE \frac{d^4 X}{dz^4} + F \frac{d^2 X}{dz^2} = 0 \quad (4.1a)$$

$$IE \frac{d^4 Y}{dz^4} + F \frac{d^2 Y}{dz^2} = 0, \quad (4.1b)$$

where  $I = \pi R^4/4$  is the moment of inertia of the rod and  $E$  is the Young modulus.

It is obvious that the system (4.1) always possesses a symmetric solution  $X = Y = 0$ .

However, we can also look for asymmetric solutions of the general form

$X = A + Bz + C \sin kz + D \cos kz$  with  $k^2 = F/EI$ , which satisfy the boundary conditions  $X = X'' = 0$  at  $z = 0$  and  $z = l$ . We find that such solutions exist,

$X = C \sin kz$ , provided  $kl = n\pi$ ;  $n = 1, \dots$ . The first such solution appears when  $F$  reaches a critical value  $F_{cr}$  given by

$$F_{cr} = \frac{\pi^2 EI}{l^2}. \quad (4.2)$$

The appearance of these solutions is already an indication of instability and, indeed, a careful study of the stability problem proves that the non-symmetric solutions correspond to lower energy. From that point Eqs. (4.1) are no longer valid, because they only apply to small deflections, and we must use the general equations of elasticity. The result is that this instability of the symmetric solution occurs for all values of  $F$  larger than  $F_{cr}$ .

What has happened to the original symmetry of the equations? It is still hidden in the sense that we cannot predict in which direction in the  $x$ - $y$  plane the rod is going to bend. They all correspond to solutions with precisely the same energy. In other words, if we apply a symmetry transformation (in this case a rotation around the  $z$ -axis) to an asymmetric solution, we obtain another asymmetric solution which is degenerate with the first one.

We call such a symmetry "spontaneously broken", and in this simple example we see all its characteristics: There exists a critical point, i.e. a critical value of some quantity (in this case the external force  $F$ ; in several physical systems it is the temperature) which determines whether spontaneous symmetry breaking will take place or not. Beyond this critical point:

- i) the symmetric solution becomes unstable;
- ii) the ground state becomes degenerate.

There exist a great variety of physical systems, both in classical and quantum physics, exhibiting spontaneous symmetry breaking, but we will not describe any other one here. The Heisenberg ferromagnet is a good example to keep in mind, because we shall often use it as a guide, but no essentially new phenomenon appears outside the ones we saw already. Therefore, we shall go directly to some field theory models.

#### 4.3 Spontaneous breaking of a global symmetry

Let  $\phi(x)$  be a complex scalar field whose dynamics is described by the Lagrangian density

$$\mathcal{L}_1 = (\partial_\mu \phi)(\partial^\mu \phi^*) - \mu^2 \phi \phi^* - \lambda (\phi \phi^*)^2, \quad (4.3)$$

where  $\mathcal{L}_1$  is a classical Lagrangian density and  $\phi$  is a classical field. No quantization is considered for the moment. The Lagrangian  $\mathcal{L}_1$  is invariant under the group  $U(1)$  of global transformations

$$\phi(x) \rightarrow e^{i\theta} \phi(x). \quad (4.4)$$

The Hamiltonian density of the system is given by

$$\mathcal{H} = (\partial_0 \phi)(\partial_0 \phi^*) + (\partial_i \phi)(\partial_i \phi^*) + V(\phi) \quad (4.5)$$

$$V(\phi) = \mu^2 \phi \phi^* + \lambda (\phi \phi^*)^2. \quad (4.6)$$

The first two terms of  $\mathcal{H}$  are positive. They can only vanish for  $\phi = \text{constant}$ . Therefore, the ground state of the system corresponds to  $\phi = \text{constant} = \text{minimum of } V(\phi)$ .  $V$  has a minimum only if  $\lambda > 0$ . In this case the position of the minimum depends on the sign of  $\mu^2$ . (Notice that we are still studying a classical field theory and  $\mu^2$  is just a parameter. It should not be taken as a "mass".)

For  $\mu^2 \geq 0$  the minimum is at  $\phi = 0$  (symmetric solution, Fig. 4a), but for  $\mu^2 < 0$  there is a whole circle of minima at the complex  $\phi$ -plane with radius  $v = (-\mu^2/2\lambda)^{1/2}$  (Fig. 4b). Any point on the circle corresponds to a spontaneous breaking of (4.4):

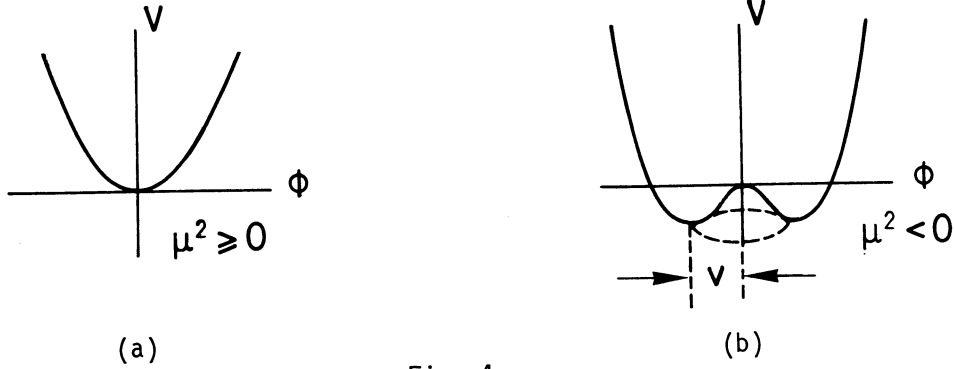


Fig. 4

We see that

- the critical point is  $\mu^2 = 0$ ;
- for  $\mu^2 \geq 0$ : the symmetric solution is stable;
- for  $\mu^2 < 0$ : spontaneous symmetry breaking occurs.

Let us assume that  $\mu^2 < 0$ . In order to reach the stable solution we translate the field  $\phi$ . It is clear that there is no loss of generality by choosing a particular point on the circle, since they are all obtained from any given one by applying the transformations (4.4). Let us, for convenience, choose the point on the real axis in the  $\phi$ -plane. We thus write

$$\phi(x) = \frac{1}{\sqrt{2}} [\nu + \psi(x) + i\chi(x)] . \quad (4.7)$$

Bringing (4.7) in (4.3) we find

$$\begin{aligned} \mathcal{L}_1(\phi) \rightarrow \mathcal{L}_2(\psi, \chi) = & \frac{1}{2} (\partial_\mu \psi)^2 + \frac{1}{2} (\partial_\mu \chi)^2 - \frac{1}{2} (2\lambda \nu^2) \psi^2 \\ & - \lambda \nu \psi (\psi^2 + \chi^2) - \frac{\lambda}{4} (\psi^2 + \chi^2)^2 . \end{aligned} \quad (4.8)$$

Notice that  $\mathcal{L}_1$  does not contain any term proportional to  $\chi^2$ .

It should be emphasized here that  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are completely equivalent Lagrangians. They both describe the dynamics of the same physical system, and a change of variables, such as (4.7), cannot change the physics. However, this equivalence is only true if we can solve the problem exactly. In this case we

shall find the same solution using either of them. However, we do not have exact solutions and we intend to apply perturbation theory, which is an approximation scheme. Then the equivalence is no longer guaranteed and, in fact, perturbation theory has much better chances to give sensible results using one language rather than the other<sup>\*)</sup>. In particular, if we use  $\mathcal{L}_1$  as a quantum field theory and we decide to apply perturbation theory taking, as the unperturbed part, the quadratic terms of  $\mathcal{L}_1$ , we immediately see that we shall get nonsense. The spectrum of the unperturbed Hamiltonian would consist of particles with negative square mass, and no perturbation corrections, at any finite order, could change that. This is essentially due to the fact that, in doing so, we are trying to calculate the quantum fluctuations around an unstable solution, and perturbation theory is just not designed to do so. On the contrary, we see that the quadratic part of  $\mathcal{L}_2$  gives a reasonable spectrum; thus we hope that perturbation theory will also give reasonable results. Therefore we conclude that our physical system, considered now as a quantum system, consists of two interacting scalar particles, one with mass  $m_\psi^2 = 2\lambda v^2$  and the other with  $m_\chi^2 = 0$ . It is the spectrum that we would have found also starting from  $\mathcal{L}_1$ , if we could solve the dynamics exactly.

The appearance of a zero-mass particle is an example of a general theorem due to Goldstone: To every generator of a spontaneously broken symmetry there corresponds a massless particle, called the Goldstone particle. This theorem is just the translation, into quantum field theory language, of the statement about the degeneracy of the ground state we saw in the previous example. The ground state of a system described by a quantum field theory is the vacuum state, and you need massless excitations in the spectrum of states in order to allow for the degeneracy of the vacuum.

Perhaps you will remember that we decided to study the phenomenon of spontaneous symmetry breaking, because we were in despair over the massless vector bosons which seemed to plague gauge theories. We see that, up to this point, our hopes were not justified. Far from arranging things, spontaneous symmetry breaking introduces its own massless particles, the Goldstone bosons, and now we have two diseases to worry about instead of one. But here will come the miracle! When combined together, the two diseases will cure each other. In order to see this remarkable phenomenon we turn to the study of the spontaneous breaking of a gauge symmetry.

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\*) This is an example of what S. Ciulli calls "broken tautology". Two different languages to describe the dynamics of a system may be completely equivalent if solved exactly (tautologies), but they may give completely different results when treated in an approximation scheme. For details, see S. Ciulli, C. Pomponiu and I.S. Stănescu: Phys. Rep. 17C, 133 (1975).



#### 4.4 Spontaneous breaking of a gauge symmetry

We consider the same model of Eq. (4.3) but we impose invariance under local U(1) transformations. According to the general rule of the previous section, this is achieved by replacing  $\partial_\mu$  by  $D_\mu$  and adding the photon kinetic energy term:

$$\mathcal{L}_1 = -\frac{1}{4} F_{\mu\nu}^2 + |(\partial_\mu + i e A_\mu)\phi|^2 - \mu\phi\phi^* - \lambda(\phi\phi^*)^2. \quad (4.9)$$

$\mathcal{L}_1$  is invariant under the gauge transformation:

$$\phi \rightarrow e^{i\theta(x)} \phi(x) \quad ; \quad A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu \theta(x). \quad (4.10)$$

The same analysis as before shows that for  $\lambda > 0$  and  $\mu^2 < 0$  there is a spontaneous breaking of the gauge symmetry. Replacing (4.7) into (4.9) we obtain:

$$\begin{aligned} \mathcal{L}_1 \rightarrow \mathcal{L}_2 = & -\frac{1}{4} F_{\mu\nu}^2 + \frac{e^2 v^2}{2} A_\mu^2 + \\ & + \frac{1}{2} (\partial_\mu \psi)^2 + \frac{1}{2} (\partial_\mu \chi)^2 - \frac{1}{2} (2\lambda v^2) \psi^2 \\ & - e v A_\mu \partial^\mu \chi + \text{coupling terms}. \end{aligned} \quad (4.11)$$

The surprising term is the second one which is proportional to  $A_\mu^2$ . It looks as though the photon has become massive! Notice that (4.11) is still gauge invariant since it is equivalent to (4.9). The gauge transformation is now obtained by replacing (4.7) into (4.10):

$$\begin{aligned} \psi(x) & \rightarrow \cos\theta(x) [\psi(x) + v] - \sin\theta(x) \chi(x) - v, \\ \chi(x) & \rightarrow \cos\theta(x) \chi(x) + \sin\theta(x) [\psi(x) + v], \\ A_\mu(x) & \rightarrow A_\mu(x) + \frac{1}{e} \partial_\mu \theta(x). \end{aligned} \quad (4.12)$$

This means that our previous conclusion, that gauge invariance forbids the presence of an  $A_\mu^2$  term, was simply wrong. Such a term can be present, only the gauge transformation is slightly more complicated; it must be accompanied by a translation of the field.

The Lagrangian (4.11), if taken as a quantum field theory, seems to describe the interaction of a massive vector particle ( $A_\mu$ ) and two scalars, one massive ( $\psi$ ) and one massless ( $\chi$ ). However, we can see immediately that something is wrong with this counting. A warning is already contained in the last term of (4.11), which is non-diagonal between  $A_\mu$  and  $\partial_\mu \chi$ . This tells us that we must be careful before interpreting the particle spectrum. A more direct way to see the trouble is to count the degrees of freedom before and after the translation:

Lagrangian (4.9) :	One massless vector field:	2	degrees
	Two scalar fields	: 2	"
	Total	: 4	"
Lagrangian (4.11):	One massive vector field :	3	"
	Two scalar fields	: 2	"
	Total	: 5	"

Since physical degrees of freedom cannot be created by a simple change of variables, we conclude that the Lagrangian (4.11) must contain fields which do not correspond to physical particles. This is indeed the case, and we can exhibit a transformation to make the unphysical fields disappear:

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2}} [v + \rho(x)] e^{i\zeta(x)/v} , \\ A_\mu(x) &= B_\mu(x) + \frac{1}{e v} \partial_\mu \zeta(x) . \end{aligned} \tag{4.13}$$

Replacing (4.13) into (4.9) we obtain:

$$\begin{aligned} \mathcal{L}_1 \rightarrow \mathcal{L}_3 &= -\frac{1}{4} B_{\mu\nu}^2 + \frac{e^2 v^2}{2} B_\mu^2 \\ &+ \frac{1}{2} (\partial_\mu \rho)^2 - \frac{1}{2} (2\lambda v^2) \rho^2 \\ &- \frac{\lambda}{4} \rho^4 + \frac{1}{2} e^2 B_\mu^2 (2v\rho + \rho^2) , \end{aligned} \tag{4.14}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu .$$

The field  $\zeta(x)$  has disappeared! Formula (4.14) describes two massive particles, a vector ( $B_\mu$ ) and a scalar ( $\rho$ ). We see that we obtained three different Lagrangians describing the same physical system.  $\mathcal{L}_1$  is invariant under the usual

gauge transformation, but it contains a negative square mass and therefore it is unsuitable for quantization.  $\mathcal{L}_2$  is still gauge invariant but the transformation laws are more complicated [Eq. (4.12)]. It can be quantized in a space containing unphysical degrees of freedom. This, by itself, is not a great obstacle and it happens frequently. For example, ordinary quantum electrodynamics is usually quantized in a space involving unphysical (longitudinal as well as scalar) photons. In fact, it is  $\mathcal{L}_2$ , in a suitable gauge, which is used for general proofs of renormalizability, as well as for practical calculations. Finally  $\mathcal{L}_3$  is no longer invariant under any kind of gauge transformations, but it exhibits clearly the particle spectrum of the theory. It contains only physical particles and they are all massive! Actually,  $\mathcal{L}_3$  can be obtained from  $\mathcal{L}_2$  by specifying the gauge of the latter.  $\mathcal{L}_3$  is non-renormalizable, by power counting, but, since it is gauge-equivalent to  $\mathcal{L}_2$  it can still be used for practical calculations. This analysis can be repeated verbatim for non-Abelian gauge theories with identical results.

The conclusion can now be stated as follows:

In a spontaneously broken gauge symmetry the gauge vector bosons acquire a mass and the would-be massless Goldstone bosons decouple and disappear. Their degrees of freedom were used in order to make possible the transition from massless to massive vector bosons. This phenomenon has been discovered by several people and it is known as the "Higgs mechanism".

This is the miracle that was announced earlier. Although we start from a gauge theory, the final spectrum contains only massive particles. And now we can state the most important theorem which really opened the way to all the wonderful applications of Yang-Mills theories:

*Theorem:* A Yang-Mills theory, spontaneously broken, although it contains massive vector bosons, remains renormalizable ('t Hooft, 1971).

## 5. MODEL BUILDING

Now that we have all the basic ingredients, we shall try to apply them to the real world and construct realistic, renormalizable models of weak and electromagnetic interactions. The essential steps of model building are quite simple:

"Do-it-yourself kit for gauge models":

- 1) Choose a gauge group  $G$ .
- 2) Choose the fields of the "elementary particles" you want to introduce, and their representations. Do not forget to include enough scalar fields to allow for the Higgs mechanism.

- 3) Write the most general renormalizable Lagrangian invariant under G. At this stage gauge invariance is still exact and all gauge vector bosons are massless.
- 4) Choose the parameters of the Higgs scalars so that spontaneous symmetry breaking occurs. In practice, this often means to choose a negative value for the parameter  $\mu^2$ .
- 5) Translate the scalars and rewrite the Lagrangian in terms of the translated fields. Choose a suitable gauge and quantize the theory.
- 6) Look at the properties of the resulting model. If it resembles physics, even remotely, publish it.
- 7) GO TO 1.

Some remarks: Gauge theories give only the general framework, not a detailed model. The latter will depend on the particular choices made in points (1) and (2). A great variety of models is possible and this partly explains the popularity of gauge theories. There is no point in describing every one which claims agreement with experiment, which is a common characteristic of most published models, see point (6). Nevertheless I shall demonstrate the efficiency of the seven-point program in a well-known example, the original model of Weinberg and Salam. I shall restrict myself, at first, to the leptonic world. The reader who is not interested in technical details is advised to skip the constructive part and go directly to the final form, after step (5) has been performed.

Step 1: We need at least four vector bosons, two charged for ordinary weak interactions, one neutral for the weak neutral currents<sup>\*)</sup> and one for the photon. It follows that the smallest possible group is  $SU(2) \times U(1)$ .

Step 2: We shall limit ourselves to the known leptons, so let us introduce the electron and its neutrino as a two-component spinor and define its left and right chiral parts:

$$\psi \equiv \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \quad L \equiv \frac{1}{2}(1+\gamma_5)\psi, \quad R \equiv \frac{1}{2}(1-\gamma_5)\psi. \quad (5.1)$$

The muon and its associated neutrino are treated in exactly the same way. Furthermore, we need the Higgs scalars which are going to break the symmetry and give masses to everybody. We learned in the last section that, for every vector boson

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\*) When the model was proposed the neutral currents had not yet been observed.

which acquires a mass, there is a corresponding scalar which becomes unphysical and decouples. Also, there is at the end at least one neutral scalar which remains physical, namely the one which we choose to translate. Since we must give masses to three vector bosons (the fourth, the photon, will remain massless), we need at least four scalars, which we are going to describe by a  $2 \times 2$  matrix  $\Phi$ . We now assign transformation properties to all these fields under the group  $SU(2) \times U(1)$ . They are given by

$$\begin{array}{ll}
 U(1) : & SU(2) : \\
 L \rightarrow e^{\frac{i}{2} g' \theta(x)} L & L \rightarrow e^{i g \vec{z} \cdot \vec{\theta}(x)} L \\
 R \rightarrow e^{\frac{i}{2} g' (\frac{1}{2} - \tau_3) \theta(x)} R & R \rightarrow R \\
 \Phi \rightarrow \Phi e^{i g' \tau_3 \theta(x)} & \Phi \rightarrow e^{i g \vec{z} \cdot \vec{\theta}(x)} \Phi
 \end{array} \tag{5.2}$$

These transformation laws are determined by simple physical requirements. For example,  $R$  must remain invariant under  $SU(2)$  otherwise we would have right-handed, charged, leptonic currents. Its transformation properties under  $U(1)$  are determined by the requirement that the photon must couple only to charged particles through a vector current. The transformation properties of  $\Phi$  will be explained below;  $g$  and  $g'$  are two arbitrary constants.

Step 3: We now write the most general, renormalizable Lagrangian invariant under (5.2). We follow the rules explained in Section 3 and we construct the corresponding covariant derivatives. The requirement of renormalizability implies that we should not introduce terms with dimensions higher than four. The result is:

$$\begin{aligned}
 \mathcal{L} = & -\frac{1}{4} \vec{A}_{\mu\nu} \cdot \vec{A}^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\
 & + \bar{L} i \gamma^\mu [ \partial_\mu - \frac{i}{2} g' B_\mu - i g \vec{z} \cdot \vec{A}_\mu ] L \\
 & + \bar{R} i \gamma^\mu [ \partial_\mu - i g' (\frac{1}{2} - \tau_3) B_\mu ] R \\
 & - \sqrt{2} \bar{L} \Phi G R - \sqrt{2} \bar{R} G \Phi^\dagger L \\
 & + \frac{1}{2} \text{Tr} | [ \partial_\mu \Phi - i g' \Phi \tau_3 B_\mu - i g \vec{z} \cdot \vec{A}_\mu \Phi ] |^2 \\
 & - \frac{1}{2} \mu^2 \text{Tr} (\Phi \Phi^\dagger) - \frac{\lambda}{4} \text{Tr} (\Phi \Phi^\dagger)^2,
 \end{aligned} \tag{5.3}$$

where  $\vec{A}_\mu$  and  $B_\mu$  are the vector gauge bosons for the groups SU(2) and U(1), respectively, and  $G$  is a  $2 \times 2$  real, diagonal, numerical matrix of the form

$$G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} . \quad (5.4)$$

The vector boson kinetic energy terms are given, as usual, by:

$$\vec{A}_{\mu\nu} \equiv \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu - g \vec{A}_\mu \times \vec{A}_\nu \quad (5.5)$$

$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu . \quad (5.6)$$

Some remarks on (5.3): i) As expected, the vector bosons  $\vec{A}_\mu$  and  $\vec{B}_\mu$  are massless. ii) L and R have different transformation properties. It follows that a fermion mass term of the form  $\bar{\psi}\psi = \bar{L}R + \bar{R}L$  is not invariant. Thus the fermions are also massless. They will acquire a mass, together with the vector bosons, via the Higgs mechanism. For this purpose the Yukawa couplings on the fourth line of (5.3) are important. The transformation laws of  $\Phi$  were determined specifically in order to allow for those terms. iii) The Lagrangian  $\mathcal{L}$  contains two coupling constants,  $g$  and  $g'$ . It has become customary to define an angle

$$\tan \theta_w \equiv g'/g . \quad (5.7)$$

Step 4: According to the analysis of the previous section we choose  $\mu^2 < 0$  in order to obtain a spontaneous symmetry-breaking solution.

Step 5: We now translate the field  $\Phi$ ,

$$\Phi \rightarrow \Phi + \frac{v}{\sqrt{2}} \mathbf{1} , \quad (5.8)$$

where  $\mathbf{1}$  denotes the unit matrix. The result of this translation is rather lengthy, but we shall single out the most interesting terms:

i) The Yukawa couplings give rise to terms:

$$-v \bar{L} G R + h.c. = -v G_1 \bar{\nu}_e \nu_e - v G_2 \bar{e} e \quad (5.9)$$

where we have used the form of  $G$  given in (5.4). We therefore see that we have generated fermion mass terms. Since we want to keep the neutrino

massless, we choose

$$G_1 = 0 \rightarrow m_\nu = 0 \quad ; \quad m_e = \nu G_2 . \quad (5.10)$$

ii) As expected, we also generate mass-terms for the vector bosons. They come from the vector boson  $\Phi$  couplings on the fifth line of (5.3) and they read:

$$\frac{1}{8} g^2 \nu^2 \vec{A}_\mu^2 + \frac{1}{8} g'^2 \nu^2 B_\mu^2 + \frac{1}{4} g g' B^\mu A_\mu^3 . \quad (5.11)$$

As we see, there exists a non-diagonal mass-matrix between the two neutral bosons,  $B_\mu$  and  $A_\mu^3$ , given by

$$M_{AB} = \frac{1}{8} \nu^2 \begin{pmatrix} g'^2 & g g' \\ g g' & g^2 \end{pmatrix} . \quad (5.12)$$

It has one eigenvalue equal to zero and the corresponding boson will be identified to the photon. By diagonalizing (5.12) we write (5.11) as:

$$\frac{1}{4} g^2 \nu^2 W_\mu^+ W_\mu^- + \frac{1}{8} \nu^2 (g^2 + g'^2) Z_\mu^2 , \quad (5.13)$$

where we have defined:

$$W_\mu^\pm = \frac{1}{\sqrt{2}} (A_\mu^1 \pm i A_\mu^2) \rightarrow m_{W^\pm} = \frac{1}{2} g \nu$$

$$Z_\mu \equiv \sin \theta_w B_\mu + \cos \theta_w A_\mu^3 \rightarrow m_Z = \frac{1}{2} \sqrt{g^2 + g'^2} \nu = \frac{m_{W^\pm}}{\cos \theta_w} \quad (5.14)$$

$$A_\mu \equiv -\cos \theta_w B_\mu + \sin \theta_w A_\mu^3 \rightarrow m_A = 0 .$$

$A_\mu$  is the photon with zero mass.  $W_\mu^\pm$  are the conventional charged intermediate vector bosons with mass  $gv/2$ , and  $Z_\mu$  is a neutral vector boson with mass  $gv/2 \cos \theta_w$ .

iii) The masses of the scalar mesons are obtained from the last line of (5.3).

If we put

$$\Phi = \frac{1}{\sqrt{2}} (\varphi \mathbb{1} + i \vec{c} \cdot \vec{\chi}) , \quad (5.15)$$

we obtain

$$m_{\psi}^2 = 2\lambda v^2 \quad ; \quad m_{\chi} = 0 . \quad (5.16)$$

The  $\psi$ -field remains physical. The three  $\chi$ -fields are massless. They correspond to the would-be Goldstone bosons and they do not represent physical particles. Their degrees of freedom were used to create the extra polarization states required to make  $W^{\pm}$  and  $Z$  massive.

iv) The couplings of the photon are the usual electromagnetic ones:

$$\frac{gg'}{\sqrt{g^2+g'^2}} \bar{e}(x) \gamma_{\mu} e(x) A^{\mu}(x) \quad (5.17)$$

which determine the electric charge  $e$  to be

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} = g \sin \theta_w . \quad (5.18)$$

The photon-charged vector boson couplings correspond to a gyromagnetic ratio equal to two.

v) The ordinary weak interactions of charged currents are of the form

$$\frac{g}{2\sqrt{2}} \bar{\nu}_e(x) \gamma^{\mu} (1+\gamma_5) e(x) W_{\mu}^{+}(x) + h.c. \quad (5.19)$$

and the Fermi coupling constant is

$$\frac{G}{\sqrt{2}} = \frac{g^2}{8m_w^2} . \quad (5.20)$$

Combining (5.14), (5.18), and (5.20) we find:

$$m_{W^{\pm}} \geq 37.5 \text{ GeV} \quad ; \quad m_Z \geq 75 \text{ GeV}, \quad (5.21)$$

which is a very striking prediction. In fact the existence of very massive intermediate bosons is a feature common to all unified theories.

vi) The neutral currents are of the form:

$$\frac{\sqrt{g^2+g'^2}}{4} \left\{ \bar{\nu} \gamma_{\mu} (1+\gamma_5) \nu - \bar{e} \gamma_{\mu} \left[ \frac{g^2-3g'^2}{g^2+g'^2} + \gamma_5 \right] e \right\} Z^{\mu} \quad (5.22)$$

Notice that they do not have a definite helicity.



There are all sorts of other couplings induced by the translation, but we will not give them here explicitly. They can be computed from (5.3). Notice only that there remain Yukawa couplings of the  $\Psi$ -meson to the leptons with strength proportional to the lepton masses. Notice also that the  $\Psi$ -mass can be arbitrarily large.

This is the simplest leptonic model which is consistent with all present-day experiments. It can be extended to a renormalizable theory (the extension requires the introduction of hadrons). Its main experimental prediction, which was brilliantly verified, was the existence of neutral currents.

## 6. EXTENSION TO HADRONS

We shall discuss here the problems connected with the extension of these ideas to the hadronic world. Since we want to use the field theory language, we are again confronted with the old question of deciding which hadrons are elementary. We therefore anticipate a certain degree of arbitrariness. We shall present all arguments in a quark model framework, but the results depend only on the underlying symmetries and not on the actual existence of physical quarks.

### 6.1 Charm

The most astonishing prediction of all gauge theories can be stated in the form of the following theorem:

The symmetry group of strong interactions is larger than  $SU(3)$ .

The proof is very simple: Let  $q$  represent the three basic quarks  $p$ ,  $n$ ,  $\lambda$ . The charged current has the familiar Cabibbo form:

$$q = \begin{pmatrix} p \\ n \\ \lambda \end{pmatrix} ; \quad h_{\mu}^{+} = \bar{q} \gamma_{\mu} (1 + \gamma_5) C^{+} q , \quad (6.1)$$

where  $C^{+}$  is a  $3 \times 3$  numerical matrix given by:

$$C^{+} = \begin{pmatrix} 0 & \cos \theta & \sin \theta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (6.2)$$

and  $\theta$  is the Cabibbo angle. The complex conjugate current  $h_\mu^-$  is formed with  $C^-$ , the transposed matrix of  $C^+$ . In a gauge theory involving  $h_\mu^+$  and  $h_\mu^-$ , the neutral current will be related to their commutator  $h_\mu^{(0)}$ , which is

$$h_\mu^{(0)} = \bar{q} \gamma_\mu (1 + \gamma_5) C^0 q \quad (6.3)$$

$$C^0 \equiv [C^+, C^-] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos^2\theta & -\cos\theta\sin\theta \\ 0 & -\cos\theta\sin\theta & -\sin^2\theta \end{pmatrix}. \quad (6.4)$$

We see that  $C^0$  has non-vanishing off-diagonal matrix elements coupled to  $\bar{n}\lambda$  or  $\bar{\lambda}n$ . This means that in a gauge theory there will be a strangeness-changing neutral current and  $\Delta S = 2$  transitions, i.e. processes such as  $K_L^0 \rightarrow \mu^+\mu^-$ , or  $K^+ \rightarrow \pi^+\nu\bar{\nu}$  with amplitudes comparable to  $K_{\mu_2}$  and  $K_{e_3}$ , as well as a  $K_1-K_2$  mass difference due to first-order weak interactions. Since such processes are definitely ruled out by experiments, we conclude that we must enlarge the symmetry of strong interactions so that the commutator (as well as anticommutator) (6.4) becomes a diagonal matrix. It turns out that this can be achieved in a variety of ways as long as the corresponding matrices are larger than  $3 \times 3$ . The detailed predictions will depend on the particular choice, but in any case, we shall have new hadrons carrying new quantum numbers. We shall call these quantum numbers collectively "charm", and it has always been emphasized that their experimental discovery would constitute the most dramatic test of these ideas. It is therefore easy to understand the general enthusiasm caused by the announcement of the new particle discoveries at Brookhaven and SLAC. Since then, the news has kept on coming in at a breathtaking pace. It is not the purpose of these lectures to review the experimental situation in detail, but we have all heard in Wiik's lectures that hadrons carrying new quantum numbers actually exist. It is exciting to think that abstract theoretical ideas, based essentially on logical and aesthetic arguments, have led to the discovery of a whole new area in particle physics.

## 6.2 $3 + 1 = SU(4)$

As an example of a larger symmetry, I shall present the  $SU(4)$  scheme. One new quark  $p'$  is introduced, whose quantum numbers are indicated in the table below:  $I$  = isospin,  $I_3$  = third component,  $Q$  = electric charge,  $S$  = strangeness,

and C = charm. The Gell-Mann Nishijima formula now reads:

$$Q = I_3 + \frac{B+S+C}{2}, \quad (6.5)$$

where B is the baryon number.

	I	I <sub>3</sub>	Q	S	C
p'	0	0	2/3	0	1
p	1/2	1/2	2/3	0	0
n	1/2	-1/2	-1/3	0	0
λ	0	0	-1/3	-1	0

The model is based on the principle of lepton-hadron symmetry. Let q and ℓ denote the quartets of quarks and leptons, respectively:

$$q = \begin{pmatrix} p' \\ p \\ n \\ \lambda \end{pmatrix}, \quad \ell = \begin{pmatrix} \nu_e \\ \nu_\mu \\ e^- \\ \mu^- \end{pmatrix}. \quad (6.6)$$

They have similar charge spectra: Q, Q, Q-1, Q-1 (Q = 2/3 for quarks and 0 for leptons), therefore, they have similar electromagnetic interactions. We postulate that they also have the same weak ones. The charged leptonic weak currents are known:

$$\ell_\mu^+ = \bar{e} \gamma_\mu (1 + \gamma_5) C_e^+ e; \quad C_\ell^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (6.7)$$

where each element of C<sub>ℓ</sub><sup>+</sup> is a 2 × 2 matrix. We thus write the same matrix for the charged hadronic current. The weak and electromagnetic Lagrangian will be of the form:

$$\begin{aligned} \mathcal{L} = & [\bar{q} \gamma_\mu Q_q q + \bar{\ell} \gamma_\mu Q_\ell \ell] A^\mu + \\ & + [\bar{q} \gamma_\mu (1+\gamma_5) C_q^+ q + \bar{\ell} \gamma_\mu (1+\gamma_5) C_\ell^+ \ell] W^\mu + \text{h.c.}, \end{aligned} \quad (6.8)$$

where  $Q_q$  and  $Q_\ell$  are the diagonal  $4 \times 4$  charge matrices of quarks and leptons. Now we introduce the quark masses, possibly via a Higgs mechanism. The mass term will be of the form  $\bar{q} M q$  with  $M$  some  $4 \times 4$  matrix. In general this matrix will not be diagonal on the basis on which we have written (6.8). The diagonalization must leave  $Q_q$  invariant and, therefore, it will change  $C_q^+$  into

$$C_q^+ = \begin{pmatrix} 0 & \nu^+ \\ 0 & 0 \end{pmatrix} \quad ; \quad \nu^+ = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}. \quad (6.9)$$

The charged hadronic current now becomes

$$\begin{aligned} h_\mu^+ &= \bar{q} \gamma_\mu (1+\gamma_5) C_q^+ q = \bar{p} \gamma_\mu (1+\gamma_5) (n \cos\theta + \lambda \sin\theta) \\ &+ \bar{p}' \gamma_\mu (1+\gamma_5) (-n \sin\theta + \lambda \cos\theta) = \\ &= \bar{p} \gamma_\mu (1+\gamma_5) n_c + \bar{p}' \gamma_\mu (1+\gamma_5) \lambda_c, \end{aligned} \quad (6.10)$$

where  $\theta$  is the Cabibbo angle and  $n_c$  and  $\lambda_c$  are the two orthogonal combinations written in (6.10). It is now straightforward to verify that

$$C_q^0 = [C_q^+, C_q^-] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad [C_q^+, C_q^-]_+ = \mathbb{1}, \quad (6.11)$$

which means that no strangeness- or charm-changing neutral currents are induced.

The introduction of the fourth quark means that the natural symmetry of strong interactions is  $SU(4)$ . This implies the existence of a large variety of "charmed" hadrons. For example, the  $0^-$  mesons, which are made out of  $q\bar{q}$  pairs, form a  $4 \times 4 = 15 + 1$  representation. Decomposed in  $SU(3)$  representations, the 15 contains the usual octet of  $C = 0$  mesons, a triplet (3) with  $C = -1$ , another one ( $\bar{3}$ ) with  $C = +1$ , and a singlet with  $C = 0$ . The  $\frac{1}{2}^+$  baryons form a

20-dimensional representation, and the same is true for the  $\frac{3}{2}^+$  resonances. Strong interactions conserve charm, so the lightest of these states decay only weakly. As you know, heavy, but weakly decaying hadrons, giving predominantly strange particles, have been observed in  $e^+e^-$  annihilations as well as in neutrino reactions. They fit nicely into the charm scheme.

An important question about these new charmed particles is the order of magnitude of their mass. Phrased differently, we would like to know how badly SU(4) is broken. In the language of the quark model, this breaking is given by the mass difference  $m_p, -m_p$ . If the order of magnitude of this difference is not restricted by the theory, the prediction of charm is not all that interesting. The remarkable feature of gauge theories is precisely that they predicted low-mass charmed hadrons, not heavier than, say, 10 GeV. Do not forget that charm was introduced in order to suppress the unwanted  $\Delta S = 1$  neutral currents and the  $\Delta S = 2$  transitions, and we can prove that this suppression is not efficient if the charmed particles are very heavy.

The proof of this statement is based on the remark that, in gauge theories, diagrams involving two intermediate W's are not of order  $G^2$ , but rather  $G\alpha$ . Let us then look at the diagram of Fig. 5a which contributes to the process  $n\bar{\lambda} \rightarrow \mu^+\mu^-$  or  $K_L^0 \rightarrow \mu^+\mu^-$ . It is of order

$$\sim \cos\theta \sin\theta \frac{g^4}{m_W^2} \sim \cos\theta \sin\theta G\alpha$$

because we assume that  $m_W$  is the largest mass around. This is disastrous because it gives a branching ratio for  $K^0 \rightarrow \mu^+\mu^-$  of the order of  $10^{-4}$ , i.e. at least four orders of magnitude too big. Fortunately there is now the diagram of Fig. 5b

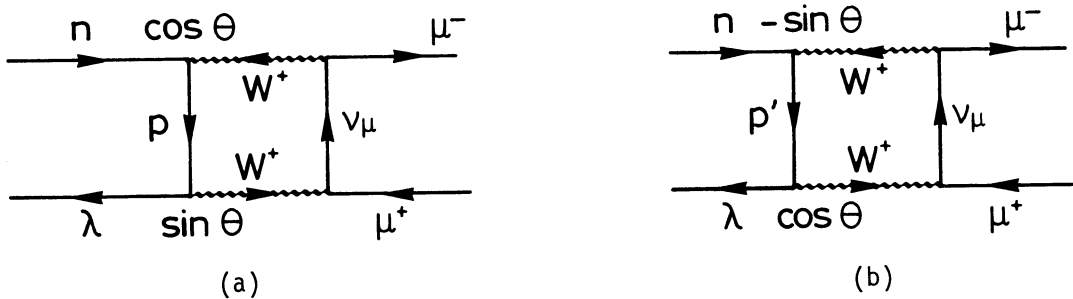


Fig. 5

which is proportional to

$$-\cos\theta\sin\theta \frac{g^4}{M_W^2} \sim -\cos\theta\sin\theta G_a .$$

The sign difference is crucial and comes from the structure of the weak current, Eq. (6.10). If SU(4) were exact, the two diagrams would cancel identically. Now their sum gives a contribution

$$\sim \cos\theta\sin\theta G_a \frac{m_P^2 - m_{P'}^2}{M_W^2} .$$

Therefore, in order to have a branching ratio of the order  $10^{-8}$ - $10^{-9}$ , we need  $\Delta m/m_W \sim 10^{-1}$ , i.e. relatively light charmed particles!

### 6.3 Colour

Charm is not the only modification which gauge theories bring to the traditional three-quark model. They also require the introduction of colour. The difference is that we had no reason to introduce charm besides the one we mentioned above, while we already needed colour outside gauge theories. There were two main reasons:

- i) The spin-statistics properties of the quarks. The introduction of a new quantum number, the colour, allows the construction of a totally antisymmetric wave function for the nucleon, without orbital angular momenta among the three quarks.
- ii) The  $\pi^0 \rightarrow 2\gamma$  lifetime. Back in 1949, J. Steinberger had calculated this decay rate assuming that  $\pi^0$  was a nucleon-antinucleon bound state.

At the one-loop level (Fig. 6), he found the correct answer. Today we believe that pions are bound states of quark-antiquark pairs, and the same calculation (due to the fractional charges of the quarks) gives an answer which is a

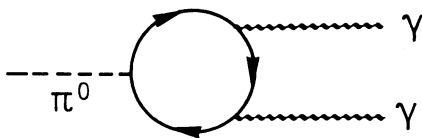


Fig. 6

factor of three too small. The introduction of the three-coloured quarks restores the right result. Let me make the logic of this argument clear: I do not mean that we want to introduce such a fundamental notion as colour based simply on the result of the lowest-order calculation. No one would take perturbation theory with respect to strong interactions so seriously. However, in this case, we have a much

stronger reason. We can show that in every theory of strong interactions which satisfies Current Algebra and PCAC, the result of the one-loop calculation is not changed by higher-order corrections. This, together with the value of R in  $e^+e^-$  collisions, makes the introduction of colour inevitable.

It is remarkable to notice that we can arrive at the same conclusion by a completely independent argument based on the renormalizability of gauge theories of weak and electromagnetic interactions. In fact, we can show that the standard Weinberg-Salam model for leptons is, strictly speaking, non-renormalizable because of the appearance of the well-known triangle anomalies (the Adler-Bell-Jackiw anomalies) in the Ward identities of the axial currents. It turns out that, in the framework of this kind of models, these anomalies cancel only if the electric charges of all elementary fermions sum up to zero. Since the known leptons have a sum equal to  $-2$  ( $e^-$ ,  $\mu^-$ ), we need precisely three colours of fractionally (Gell-Mann/Zweig) or integrally (Han-Nambu) charged quarks, including the charmed ones, in order to satisfy the no-anomaly condition.

#### 6.4 The "Standard" model (C. Bouchiat, J. Iliopoulos and Ph. Meyer)

This is the extension of the Weinberg-Salam model to the hadronic world. This extension is completely straightforward. We have four quartets of elementary fermions; the leptons

$\nu_e$	$p'$	$p'$	$p'$
$\nu_\mu$	$p$	$p$	$p$
$e^-$	$n$	$n$	$n$
$\mu^-$	$\lambda$	$\lambda$	$\lambda$
	blue	white	red

and the three-coloured quarks, and all are treated identically. Thus we form doublets:

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}; \begin{pmatrix} p \\ n_c \end{pmatrix}, \begin{pmatrix} p' \\ \lambda_c \end{pmatrix}$$

blue, white, red

with  $n_c = n \cos \theta + \lambda \sin \theta$ ,  $\lambda_c = -n \sin \theta + \lambda \cos \theta$ , and for each doublet (there are eight of them) we repeat the construction of Section 5. Notice that in SU(4) all quarks participate in weak interactions, while in SU(3) the  $\lambda_c$  quark does not.

We can try to go beyond this standard model and give real meaning to the "lepton-hadron symmetry" by putting leptons and quarks in the same representation. There exist already several such models, but there is no time to go into them now. They usually tend to violate the separate conservation of baryon and lepton numbers, and we need super-heavy W's -- sometimes as heavy as one gramme! -- in order to suppress unwanted transitions such as proton decays. Their attractive feature is that, sometimes, they suggest an intimate connection between gauge theories and gravity.

## 7. STRONG INTERACTIONS

### 7.1 Are strong interactions simple?

I believe that by now everybody is convinced that non-Abelian gauge theories describe the weak and electromagnetic interactions in terms of a renormalizable field theory. This, by itself, represents enormous progress compared to the phenomenological description that we used to have. However, the impact of gauge theories was in fact much deeper and they have considerably changed our way of thinking. For example, each one of us has often tried to explain elementary particle physics to non-specialists, and a standard way of starting was to talk about the different interactions. We were then inclined to write, schematically, the Lagrangian of the world as a sum:

$$\mathcal{L} = \mathcal{L}_{\text{strong}} + \mathcal{L}_{\text{em}} + \mathcal{L}_{\text{weak}} + \dots, \quad (7.1)$$

where the different pieces were supposed to be independent from one another. With gauge theories this is no more possible. The reason is that non-Abelian gauge theories, unlike  $\phi^4$  or Yukawa interactions, are only renormalizable if every term in the Lagrangian, no matter what its relative strength, respects the Ward identities. Such a requirement severely restricts the possible forms of all interactions including the strong ones. The ultimate goal is to find a non-Abelian gauge theory for which  $\mathcal{L}$  is so restricted that all symmetries of strong interactions arise naturally. Obviously, I do not have this ultimate model to hand, but several investigations of its possible properties tend to suggest that the domain of non-Abelian gauge theories covers in fact all elementary particle physics. This may sound strange since we were told several times that field theory, at least in the form of renormalized perturbation theory, is not applicable to strong interactions. However, I would like to remind you of a remarkable evolution which took place slowly over the last years.



When in the past we were trying to discover the possible forms of strong interactions, we were using, most of the time, the results from hadronic collisions. The resulting picture invariably appeared to be too complicated to allow for a simple interpretation. We have by now good reasons to believe that this complexity should not be attributed to the fundamental interactions themselves, but is instead due to the fact that the objects we are dealing with, namely the hadrons, are themselves too complicated. It is as if we were trying to discover quantum electrodynamics by studying the interactions among complex molecules. The great progress made in the last decade was the realization that the strong interactions look completely different, in fact much simpler, if we study the processes in which the different current operators are involved. We thus arrived at the conclusion that, as far as the basic strong interactions are concerned, the currents may be more fundamental objects than the hadrons.

This idea grew gradually through the successes of Current Algebra and the surprising results from the deep inelastic lepton-nucleon scattering and  $e^+e^-$  annihilation. The simplicity of strong interactions in these reactions is expressed by the parton model, which assumes that the target nucleon is made out of an assembly of "elementary" constituents which interact with the incident leptonic current as free, point-like charges. Indeed, strong interactions have become too simple; it is as if they did not exist! As Gell-Mann said: "... in deep inelastic scattering Nature reads only free-field theory books". The theoretical question now is, How can the partons be so tightly bound in order to form a nucleon and still act like free particles in deep inelastic collisions? Stated differently, the same question is, Under which conditions may a fully interacting field theory simulate a free-field theory behaviour?

In order to answer these questions let us look once more at the experimental results. In deep inelastic scattering we measure the structure functions  $F_i$ . We would expect them to depend on two variables,  $Q^2$  and  $\nu$ , where  $Q$  is the momentum transfer among the leptons and  $\nu = 2 pQ$ ,  $p$  being the momentum of the target nucleon. Instead we find that when both  $Q^2$  and  $\nu$  become large with fixed ratio  $x = Q^2/\nu$ , the  $F_i$ 's depend only on  $x$ . This result is very interesting, precisely because it is very easy to understand it by using a naïve (and wrong!) reasoning: The  $F_i$ 's are dimensionless. Therefore they can only depend on dimensionless variables:  $F_i(x, M^2/Q^2)$ , where  $M^2$  is some characteristic mass (e.g. the nucleon mass). When  $Q^2 \rightarrow \infty$  with fixed  $x$ ,  $M^2/Q^2 \rightarrow 0$  and we are left with only the  $x$ -dependence. In other words, this argument states the intuitive idea that at very high energies the masses are unimportant.

This is a very simple argument, unfortunately it is also a wrong one! Everybody who has ever calculated a one-loop Feynman diagram knows that the result often

contains factors of the form  $\ln(M^2/Q^2)$ , and consequently the amplitude is not an analytic function of the masses. We see, however, that the answer to our questions lies in a better understanding of the mass-dependence of the amplitudes in field theory. This dependence is studied using the Callan-Symanzik equation.

## 7.2 The Callan-Symanzik equation

Let us consider the simplest renormalizable field theory in four dimensions, which is the self-interacting, neutral, scalar field:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_0(x))^2 - \frac{1}{2} \mu_0^2 \phi_0^2(x) - \frac{g_0}{4!} \phi_0^4(x), \quad (7.2)$$

where the subscript 0 indicates that all these quantities are unrenormalized. We know that Green functions calculated from (7.2) are often divergent, but they can be defined if we introduce a suitable cut-off which we shall call  $\Lambda$ . For example, we can imagine that we cut all loop integrations at momenta of the order of  $\Lambda$ .

In this case the unrenormalized Green functions will depend on:

$$\Gamma_0^{(2n)}(p_1, \dots, p_{2n}; \mu_0, g_0, \Lambda), \quad (7.3)$$

where  $\Gamma_0^{(2n)}$  is the unrenormalized Green function with  $2n$  external lines carrying momenta  $p_1, \dots, p_{2n}$  <sup>\*)</sup>. Since we are interested in the mass dependence, it is natural to try to compute the derivative of  $\Gamma_0^{(2n)}$  with respect to  $\mu_0$ . This is relatively easy because the only dependence of  $\Gamma_0$  on  $\mu_0$  comes from the different propagators of the form  $(k^2 - \mu_0^2)^{-1}$ . But since I shall never use the explicit form of this derivative, let me just define a new function  $\hat{\Gamma}_0^{(2n)}$  by

$$\frac{\partial}{\partial \mu_0^2} \Gamma_0^{(2n)}(p_1, \dots, p_{2n}; \mu_0, g_0, \Lambda) \equiv \hat{\Gamma}_0^{(2n)}(p_1, \dots, p_{2n}; \mu_0, g_0, \Lambda). \quad (7.4)$$

Until now we have not used the fact that the theory is renormalizable. This means that there exists a well-defined procedure that allows us to take the limit  $\Lambda \rightarrow \infty$  and obtain the renormalized Green functions, which I shall denote by  $\Gamma^{(2n)}$ . For the  $\phi^4$  theory this procedure requires the introduction in (7.2) of counter-terms which will renormalize the mass, the coupling constant, and the wave function of the field. Consequently we obtain  $\Gamma^{(2n)}$  from  $\Gamma_0^{(2n)}$  by replacing the bare

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\*) For technical simplicity we shall consider  $\Gamma_0^{(2n)}$  to be the so-called "one-particle irreducible" (1-PI) Green function which is defined as the sum of all Feynman diagrams with  $2n$  external lines which cannot be separated into two disconnected diagrams by cutting only one internal line.

quantities  $\mu_0$  and  $g_0$  by their renormalized values  $\mu$  and  $g$ , and by multiplying each external line by the wave function renormalization which we call  $Z_3^{1/2}$ . In other words, the fact that (7.2) is a renormalizable theory implies the existence of functions:

$$\mu(\mu_0, g_0, \Lambda) ; g(\mu_0, g_0, \Lambda) ; Z_3(\mu_0, g_0, \Lambda) , \quad (7.5)$$

such that

$$\Gamma_0^{(2n)}(p_1, \dots, p_{2n}; \mu_0, g_0, \Lambda) = Z_3^n(\mu_0, g_0, \Lambda) \Gamma^{(2n)}(p_1, \dots, p_{2n}; \mu, g) + O(1/\ln \Lambda) , \quad (7.6)$$

where  $O(1/\ln \Lambda)$  stands for terms which vanish in the limit  $\Lambda \rightarrow \infty$ . In the same way we can write for  $\hat{\Gamma}_0^{(2n)}$ :

$$\hat{\Gamma}_0^{(2n)}(p_1, \dots, p_{2n}; \mu_0, g_0, \Lambda) = Z(\mu_0, g_0, \Lambda) Z_3^n(\mu_0, g_0, \Lambda) \hat{\Gamma}^{(2n)}(p_1, \dots, p_{2n}; \mu, g) + O(1/\ln \Lambda) , \quad (7.7)$$

where  $\hat{\Gamma}^{(2n)}$  is the corresponding renormalized function and  $Z$  is another function of the form (7.5). Notice that the renormalized Green functions  $\Gamma^{(2n)}$  and  $\hat{\Gamma}^{(2n)}$  are independent of  $\Lambda$ ;  $\mu$  and  $g$  are the physical values of the mass and the coupling constant, respectively.

The Callan-Symanzik equation is now obtained by combining (7.4), (7.6), and (7.7), and using the chain rule of differentiation:

$$\begin{aligned} Z \cdot Z_3^n \cdot \hat{\Gamma}^{(2n)} &= \hat{\Gamma}_0^{(2n)} = \frac{\partial}{\partial \mu_0^2} \Gamma_0^{(2n)} = \frac{\partial}{\partial \mu_0^2} [Z_3^n \Gamma^{(2n)}] \\ &= n \frac{\partial \ln Z_3}{\partial \mu_0^2} Z_3^n \Gamma^{(2n)} + Z_3^n \left[ \frac{\partial \mu^2}{\partial \mu_0^2} \frac{\partial}{\partial \mu^2} + \frac{\partial g}{\partial \mu_0^2} \frac{\partial}{\partial g} \right] \Gamma^{(2n)} . \end{aligned} \quad (7.8)$$

This equation can also be written as

$$\begin{aligned} & \left[ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + n\gamma(g) \right] \Gamma^{(2n)}(p_1, \dots, p_{2n}; \mu, g) = \\ & = \mu^2 \delta(g) \hat{\Gamma}^{(2n)}(p_1, \dots, p_{2n}; \mu, g), \end{aligned} \quad (7.9)$$

where

$$\beta(g) \equiv \frac{\mu^2 (\partial g / \partial \mu^2)}{2 (\partial \mu^2 / \partial \mu^2)}, \quad (7.10)$$

$$\gamma(g) \equiv \frac{\mu^2 (\partial \ln Z_3 / \partial \mu^2)}{2 (\partial \mu^2 / \partial \mu^2)}, \quad (7.11)$$

$$\delta(g) \equiv \frac{Z}{2 (\partial \mu^2 / \partial \mu^2)}. \quad (7.12)$$

It is easy to show that the functions  $\beta$ ,  $\gamma$ , and  $\delta$  approach well-defined limits when  $\Lambda \rightarrow \infty$  and, since they are dimensionless, they can only depend on  $g$ . Equation (7.9) is the Callan-Symanzik equation which involves only renormalized quantities.

### 7.3 The deep Euclidean region

The Euclidean region is the one in which all momenta  $p_1, \dots, p_{2n}$  are Euclidean, i.e. they have real space parts and imaginary time parts ( $p_i^2 < 0$ ). Let us now multiply all the  $p_i$ 's with a real parameter  $\lambda$ :  $\Gamma^{(2n)}(\lambda p_1, \dots, \lambda p_{2n}; \mu, g)$ . The deep Euclidean region is reached by choosing  $\lambda$  very large, while keeping any partial sum among different  $p_i$ 's different from zero (except, of course, the trivial sum of all momenta which vanishes from energy momentum conservation). In other words, in the deep Euclidean region all masses, as well as all momentum transfers, become large and negative. In that sense it is the most unphysical region. Nevertheless, we shall see that the study of Green functions in that region presents a great physical interest.

The first step is to simplify the Callan-Symanzik equation by using a theorem, due originally to Weinberg. It states that, to every order of perturbation theory,  $\hat{\Gamma}^{(2n)}$  is negligible compared to  $\Gamma^{(2n)}$  by at least a power of  $\lambda^{-1}$

when  $\lambda \rightarrow \infty$ . Thus, in this limit, the equation becomes

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + n\gamma(g) \right] \Gamma_{as}^{(2n)}(\lambda p_1, \dots, \lambda p_n; \mu, g) = 0, \quad (7.13)$$

where  $\Gamma_{as}^{(2n)}$  denotes the asymptotic form of  $\Gamma^{(2n)}$  when  $\lambda$  gets large.

We can still simplify (7.13) by ordinary dimensional analysis:  $\Gamma^{(2n)}$  has dimensions  $\mu^{4-2n}$  and, therefore, it can be written as

$$\Gamma^{(2n)}(\lambda p_1, \dots, \lambda p_n; \mu, g) \equiv \mu^{4-2n} F^{(2n)}\left(\frac{\lambda p_1}{\mu}, \dots, \frac{\lambda p_n}{\mu}; g\right) \quad (7.14)$$

Let us now define a new function:

$$\Phi^{(2n)}(\lambda p_1, \dots, \lambda p_n; \mu, g) \equiv \lambda^{2n-4} \Gamma^{(2n)} = \left(\frac{\mu}{\lambda}\right)^{4-2n} F^{(2n)}\left(\frac{\lambda p_1}{\mu}, \dots, \frac{\lambda p_n}{\mu}; g\right) \quad (7.15)$$

By definition,  $\Phi_{as}^{(2n)}$  satisfies (7.13) and is a function of  $\lambda/\mu$ . Therefore

$$\mu \frac{\partial}{\partial \mu} \Phi^{(2n)} = -\lambda \frac{\partial}{\partial \lambda} \Phi^{(2n)}. \quad (7.16)$$

Using (7.16) we can trade the mass derivative in (7.13) for one with respect to  $\lambda$ , which is the parameter that scales all momenta:

$$\left[ -\lambda \frac{\partial}{\partial \lambda} + \beta(g) \frac{\partial}{\partial g} + n\gamma(g) \right] \Phi_{as}^{(2n)}(\lambda p_1, \dots, \lambda p_n; \mu, g) = 0. \quad (7.17)$$

(7.17) is the desired form of the equation.

#### 7.4 Solution of the Callan-Symanzik equation

Equation (7.17) can be solved by using Monge's standard method. We change variables from  $(\lambda, g)$  to  $(\lambda, \bar{g})$ , where  $\bar{g}$  is a function of  $\lambda$  and  $g$  satisfying

$$\left[ -\lambda \frac{\partial}{\partial \lambda} + \beta(g) \frac{\partial}{\partial g} \right] \bar{g}(\lambda, g) = 0 \quad (7.18)$$

with the boundary condition

$$\bar{g}(1, g) = g. \quad (7.19)$$

Equation (7.18) is equivalent to

$$\lambda \frac{\partial \bar{g}}{\partial \lambda} = \beta(g). \quad (7.20)$$

The general solution of (7.17) is now given by

$$\Phi^{(2n)}(\lambda p_1, \dots, \lambda p_{2n}; \mu, g) = \Phi^{(2n)}(p_1, \dots, p_{2n}; \mu, \bar{g}) \exp\left\{n \int_1^{\lambda} d\lambda' \gamma[\bar{g}(\lambda', g)]\right\} \quad (7.21)$$

The physical meaning of (7.21) is clear. Scaling all momenta of a Green function by a common factor  $\lambda$  and taking  $\lambda$  large, has the following effects: i) it multiplies every external line by a factor (the exponential of 7.21), and ii) it replaces the physical coupling constant  $g$  by an effective one  $\bar{g}$  which is the solution of (7.20). In other words, in the deep Euclidean region, the effective strength of the interaction is not determined by the physical coupling constant  $g$ , but rather by  $\bar{g}$ .

### 7.5 Asymptotic freedom

The fact that the coupling constant of a renormalizable field theory depends on the external momenta can be understood by a simple classical argument. Let us take electrodynamics. The coupling constant is the electric charge. Its magnitude is measured by its effects on surrounding charges. Let us assume that we have a charge  $+Q$  inside a polarizable medium, for example water. We bring close to it another charge  $-Q$ . The water molecules between the two are polarized by the electric field and they tend to screen the two charges. The net result is that the charge  $-Q$  sees an effective value of the positive charge, which depends on the distance between them. At large distances (small momenta) the effect of the screening is very important and the effective value of the charge tends to zero. At small distances (large momenta), on the other hand, this value gets larger. In a quantum language the same effect occurs also in the vacuum because of vacuum polarization. Equation (7.20) allows a complete study of this phenomenon.

Let us assume that, for  $\lambda = 1$ , we start with a coupling constant equal to  $g$ . Then the condition (7.19) tells us that  $\bar{g}(1,g) = g$ . We vary  $\lambda$ ;  $\bar{g}$  varies and, in general,  $\bar{g}(\lambda,g) \neq g$ . Equation (7.20) shows that if  $\beta > 0$ ,  $\bar{g}$  increases with increasing  $\lambda$ , and it will continue to increase as long as  $\beta$  remains positive. The limit of  $\bar{g}$  when  $\lambda \rightarrow \infty$  will be the first zero of  $\beta$  on the right of the initial value  $g$ . If  $\beta(x)$  has no zeros for  $x > g$ , then  $\bar{g} \rightarrow \infty$  for  $\lambda \rightarrow \infty$ . Now let us take the case when  $\beta(g) < 0$ . Then  $\bar{g}$  decreases with increasing  $\lambda$  and  $\lim_{\lambda \rightarrow \infty} \bar{g}(\lambda,g) =$  first zero of  $\beta(x)$  for  $x < g$ . Finally, if  $\beta(g) = 0$ ,  $\partial \bar{g} / \partial \lambda = 0$ , and  $\bar{g}$  is independent of  $\lambda$ .

This analysis shows that we can classify the zeros of  $\beta$  in two classes: Those

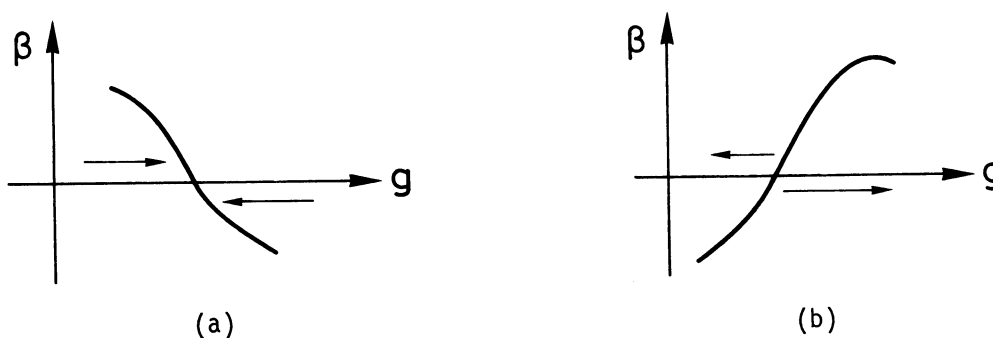


Fig. 7

of Fig. 7a are "attractors", i.e. if we start somewhere in their neighbourhood,  $\bar{g}$  approaches them for  $\lambda \rightarrow \infty$ . Those of Fig. 7b are "repulsors", i.e.  $\bar{g}$  goes further away from them when  $\lambda \rightarrow \infty$ . An attractor is always followed by a repulsor (multiple zeros must be counted accordingly). The conclusion is that the asymptotic behaviour of a field theory depends on the position and nature of the zeros of the function  $\beta$ .

As long as perturbation theory is our only guide, we cannot say anything about the properties of  $\beta(x)$  for arbitrary  $x$ . We do not know whether it has any zeros, let alone their nature. The only information that perturbation theory can hopefully provide is the behaviour of  $\beta(x)$  at the vicinity of  $x = 0$ . We know that  $\beta(0) = 0$  because  $g = 0$  is a free field theory. The nature of this zero (attractor or repulsor) will depend on the sign of the first non-vanishing term in the expansion of  $\beta(g)$  in powers of  $g$ . But this expansion is precisely perturbation theory. Therefore the properties of the zero of the  $\beta$ -function at the origin can be extracted from perturbation. If  $\beta$  starts as in Fig. 8a, i.e. from positive values, the origin is a repulsor. The effective coupling constant will be driven away to larger values as we go deeper and deeper into the deep Euclidean region. On the contrary, if the first term of  $\beta$  is negative the origin is an

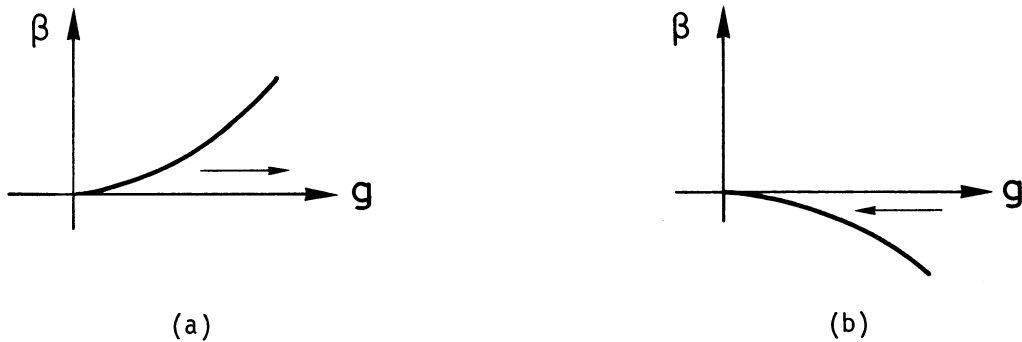


Fig. 8

attractor. If we start somewhere between the origin and the next zero of  $\beta$ , the effective coupling constant will become smaller and smaller and it will vanish in the limit. Such a theory is called "asymptotically free".

And now we shall state the following, very important theorem:

Out of all renormalizable field theories, only the non-Abelian gauge theories are asymptotically free.

This theorem is proven simply by exhaustion. We calculate the  $\beta$ -function in the one-loop approximation for  $\phi^4$ , Yukawa, QED, and non-Abelian gauge theories. Only the latter have  $\beta$  negative.

## 7.6 Physical applications

Until now we have worked into the deep Euclidean region, which is an unphysical one. We need a careful and rather technical analysis, in order to show that these considerations apply also to the asymptotic region of  $e^+e^-$  annihilation and that of deep inelastic lepton-hadron scattering. They do not apply, at least not in a straightforward way, to ordinary hadronic collisions. We shall skip this part and go directly to the physical consequences.

The message of the last theorem is clear: If we want to understand the success of the naïve parton model in a field theory language, we must assume that strong interactions are described by non-Abelian gauge theories. In the deep inelastic region, asymptotic freedom has already set in and the effective strength of strong interactions has become very small. In the opposite limit, namely at very small external momenta, Eq. (7.20) shows that the effective coupling constant increases, and it may even tend to infinity if  $\beta(x)$  has no zeros for  $x > 0$ . We could not dream of a more convenient field theory to describe strong interactions. It has all the desired properties: asymptotic freedom and infrared slavery, vanishingly small coupling constant at very short distances and infinitely large at very large ones. It gives us a framework for understanding at the same time



the free character of the quarks when probed in the deep inelastic region and their strong interactions in ordinary experiments. Furthermore, we hope that their infinitely strong coupling constants at large distances may provide the mechanism for the permanent confinement of the partons (quarks as well as non-Abelian gauge gluons) inside the hadrons.

There are several ways of realizing a non-Abelian gauge theory for strong interactions, depending on the quark model that has been assumed. Among them there is one which is particularly simple and is favoured by most theorists. It uses the Gell-Mann/Zweig fractionally charged quarks in three colours and four species (including charm), and it assumes that the gauge group of strong interactions is the group  $SU(3)$  of the colour, namely the one that mixes the three columns of blue, white, and red quarks leaving the rows unchanged. We often denote this group by  $SU(3)'$  in order to distinguish it from the ordinary  $SU(3)$ , which mixes the last three rows of the quarks, leaving the columns, as well as charm, unchanged.

The most important consequence of this scheme is that it allows us to calculate the expected violations of scaling in deep inelastic experiments. The basic assumption is that we have already reached sufficiently high energies, so that the effective coupling constant of strong interactions is small and we can use the results of low-order perturbation theory. We can thus predict the scaling violations at FNAL or SPS energies by using, as input, the data from SLAC. For example, the observed large  $y$ -anomalies in antineutrino reactions at FNAL have been fitted recently by using this standard model. It is too early yet to tell whether such fits are successful, but the important point is that the hypothesis of asymptotic freedom gives well-defined predictions which can be tested experimentally. With precise measurements ( $\sim 1-5\%$ ) of the structure functions of deep inelastic  $\mu$  or  $\nu$  scattering at FNAL or SPS, for different values of  $x$  and  $Q^2$ , we can: i) test the idea of asymptotic freedom experimentally and, if the answer is positive, ii) determine the gauge group of strong interactions.

## 8. CONCLUSION

We tried to combine all available experimental results from all processes involving currents, at low energies as well as in the deep inelastic region, and we came to the conclusion that a consistent picture arises if we postulate that all interactions among elementary particles, from the strong down to the gravitational ones, are described by non-Abelian gauge theories. We are thus free to speculate on possible ways of obtaining a really unified picture in which all of them will be different manifestations of a single fundamental interaction. And these speculations are no longer in the domain of science fiction, but in that of

serious scientific investigation. All this enormous progress in our understanding of elementary particles was made during these last years. Using a familiar expression, I would say that we have been through great vintage years. They were certainly the most exciting I can possibly remember and, according to several of my more experienced colleagues, they were the most exciting for a long time. Both theory and experiments made fantastic progress, which was unprecedented in recent history. And, more important, the progress was parallel and complementary. Theoretical ideas were initiating successful experiments, and experimental results were stimulating further theoretical work. We have to go back many years, at least as far as the discovery of parity violation, in order to find a similar fruitful cooperation. But I believe that today's results will prove to be more fundamental and far-reaching. Nor are we yet at an end. We are still actively involved in exciting theoretical as well as experimental work and we are looking forward confidently to even greater discoveries.

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